NOTE ON FRAMES AND THE ADJOINT REPRESENTATION

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1. Transformations as Changes-of-Frame

Let G be a Lie group acting on a manifold M, with $g \in G$ a transformation. We can view g as a **change of frames** from a frame $\mathfrak{a}$ to a frame $\mathfrak{b}$: write $g = (\mathfrak{b} \leftarrow \mathfrak{a})$. Here a **frame** is a way of writing elements of M in coordinates, and "changing frames" means changing the expression of points from one coordinate system to another.

Explicitly, given a point $P \in M$, and a frame $\mathfrak{a}$, we can express P in $\mathfrak{a}$'s coordinate system:

$$P_{\mathfrak{a}} = \begin{bmatrix} P_{\mathfrak{a}}^1 \\ \vdots \\ P_{\mathfrak{a}}^d \end{bmatrix} \in \mathbb{R}^d.$$

The transformation $g = (\mathfrak{b} \leftarrow \mathfrak{a})$ changes P's representation to $\mathfrak{b}$ coordinates:

$$P_{\mathfrak{b}} = (\mathfrak{b} \leftarrow \mathfrak{a}) P_{\mathfrak{a}}.$$

Changes of frame compose and invert like you would expect:

$$(\mathfrak{c} \leftarrow \mathfrak{b})(\mathfrak{b} \leftarrow \mathfrak{a}) = (\mathfrak{c} \leftarrow \mathfrak{a})$$

and

$$(\mathfrak{b} \leftarrow \mathfrak{a})^{-1} = (\mathfrak{a} \leftarrow \mathfrak{b}).$$

2. Tangent Vectors in Frames

Like points in M, vectors in the Lie algebra T_eG are expressed in frames. This is because a vector $X \in T_eG$ is viewed as a transformation which is "infinitesimally close" to the identity, which means that for any frame $\mathfrak{a}$, elements of the Lie algebra can be viewed as tiny perturbations of the frame $\mathfrak{a}$. If a Lie algebra vector X is expressed in frame $\mathfrak{a}$, we write $X^{\mathfrak{a}}$, and interpret $\exp(X^{\mathfrak{a}}) \in G$ as a perturbation of frame $\mathfrak{a}$, denoted $(\mathfrak{a}' \leftarrow \mathfrak{a})$ or $(\mathfrak{a} \leftarrow \mathfrak{a}')$.

Which frame X is expressed in tells you which side of the transform you can apply it to: if $g = (\mathfrak{b} \leftarrow \mathfrak{a})$, you must apply $\mathfrak{b}$-frame tangent vectors on the left:

$$\begin{aligned} X^{\mathfrak{b}} \oplus (\mathfrak{b} \leftarrow \mathfrak{a}) &= \exp(X^{\mathfrak{b}})(\mathfrak{b} \leftarrow \mathfrak{a}) \\ &= (\mathfrak{b}' \leftarrow \mathfrak{b})(\mathfrak{b} \leftarrow \mathfrak{a}) \\ &= (\mathfrak{b}' \leftarrow \mathfrak{a}). \end{aligned}$$

so that the perturbed transformation's "type" $(\mathfrak{b} \leftarrow \mathfrak{a})$ is preserved. Left-$\ominus$ works similarly, producing a tangent vector in the output frame:

$$\begin{aligned}(\mathfrak{b}' \leftarrow \mathfrak{a}) \ominus (\mathfrak{b} \leftarrow \mathfrak{a}) &= \log\left((\mathfrak{b}' \leftarrow \mathfrak{a})(\mathfrak{a} \leftarrow \mathfrak{b})\right) \\ &= \log(\mathfrak{b}' \leftarrow \mathfrak{b}) \\ &= X^{\mathfrak{b}} \in T_eG.\end{aligned}$$

Similarly, you apply $\mathfrak{a}$-frame tangent vectors on the right:

$$\begin{aligned}(\mathfrak{b} \leftarrow \mathfrak{a}) \oplus X^{\mathfrak{a}} &= (\mathfrak{b} \leftarrow \mathfrak{a}) \exp(X^{\mathfrak{a}}) \\ &= (\mathfrak{b} \leftarrow \mathfrak{a})(\mathfrak{a} \leftarrow \mathfrak{a}') \\ &= (\mathfrak{b} \leftarrow \mathfrak{a}').\end{aligned}$$

And

$$\begin{aligned}(\mathfrak{b} \leftarrow \mathfrak{a}') \ominus (\mathfrak{b} \leftarrow \mathfrak{a}) &= \log\left((\mathfrak{a} \leftarrow \mathfrak{b})(\mathfrak{b} \leftarrow \mathfrak{a}')\right) \\ &= \log(\mathfrak{a} \leftarrow \mathfrak{a}') \\ &= X^{\mathfrak{a}} \in T_eG.\end{aligned}$$

3. The Adjoint

The **adjoint** is the mechanism by which we change the frame of tangent vectors. Particularly, if $X^{\mathfrak{a}} \in T_eG$ is a tangent vector expressed in $\mathfrak{a}$, and $\mathfrak{b}$ is any other frame, then

$$X^{\mathfrak{b}} = \mathrm{Ad}_{\mathfrak{b} \leftarrow \mathfrak{a}} X^{\mathfrak{a}}.$$

So what does the adjoint mapping actually look like? If we are viewing the group elements and the tangent vectors correspondingly as matrices, then its form is very simple: We view $X^{\mathfrak{a}}$ as a small perturbation of frame $\mathfrak{a}$, and so to obtain a frame-$\mathfrak{b}$ perturbation, then we have to do the following:

- Convert incoming points from $\mathfrak{b}$ to $\mathfrak{a}$.
- Apply the infinitesimal perturbation in $\mathfrak{a}$.
- Convert the outgoing points back from $\mathfrak{a}$ to $\mathfrak{b}$.

Mathematically, this is simply **conjugation**:

$$X^{\mathfrak{b}} = (\mathfrak{b} \leftarrow \mathfrak{a}) X^{\mathfrak{a}} (\mathfrak{a} \leftarrow \mathfrak{b}).$$

So, for a given $g = (\mathfrak{b} \leftarrow \mathfrak{a}) \in G$, the adjoint can be defined as the linear conjugation mapping:

$$\mathrm{Ad}_{\mathfrak{b} \leftarrow \mathfrak{a}} : X^{\mathfrak{a}} \mapsto (\mathfrak{b} \leftarrow \mathfrak{a}) X^{\mathfrak{a}} (\mathfrak{a} \leftarrow \mathfrak{b}).$$

Or, without the "frame" notation:

$$\mathrm{Ad}_g : X \mapsto gXg^{-1}.$$

If we're considering G as an abstract Lie group (not represented as matrices), then we can't directly multiply group elements and tangent vectors, but we can still define Ad implicitly using exp:

$$\exp(\mathrm{Ad}_g X) = g \exp(X) g^{-1},$$

or, equivalently,

$$\exp(\mathrm{Ad}_g X) g = g \exp(X).$$

Note that this captures the fact that Ad_g moves perturbations from input to output frame (and vice-versa).