# NOTE ON THE DERIVATIVE OF THE GROUP ACTION

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## 1. General Formula

Consider a transformation $(\mathfrak{b} \leftarrow \mathfrak{a})$ and a point $P$, so that $P_\mathfrak{b} = (\mathfrak{b} \leftarrow \mathfrak{a})P_\mathfrak{a}$. Our goal is to find an expression for the derivative of $P_\mathfrak{b}$ with respect to a perturbation of the frame $\mathfrak{b}$. To find this expression we should apply a perturbation $\exp(X^\mathfrak{b})$ on the left, and map it through to an additive perturbation of $P_\mathfrak{b}$. Concretely, let $A_P(\mathfrak{b} \leftarrow \mathfrak{a}) = P_\mathfrak{b}$, and let $D_\mathfrak{b} A_P : T_e G \to \mathbb{R}^d$ be the derivative as described. Then we should be able to write

$$(D_\mathfrak{b} A_P)(X^\mathfrak{b}) + P_\mathfrak{b} \approx (X^\mathfrak{b} \oplus (\mathfrak{b} \leftarrow \mathfrak{a}))P_\mathfrak{a},$$

where $\approx$ is first-order equality (i.e. equality assuming that the squares of tangent vectors are zero).

It's most convenient to do this calculation by viewing transformations and tangent vectors in their matrix representation, so that $X^\mathfrak{b}$ and $(\mathfrak{b} \leftarrow \mathfrak{a})$ are $d \times d$ matrices. In this representation,

$$\begin{aligned} \exp(X^\mathfrak{b}) &= I + X^\mathfrak{b} + \frac{1}{2}(X^\mathfrak{b})^2 + \frac{1}{3!}(X^\mathfrak{b})^3 + \cdots \\ &\approx I + X^\mathfrak{b}. \end{aligned}$$

Hence

$$\begin{aligned} (X^\mathfrak{b} \oplus (\mathfrak{b} \leftarrow \mathfrak{a}))P_\mathfrak{a} &= \exp(X^\mathfrak{b})(\mathfrak{b} \leftarrow \mathfrak{a})P_\mathfrak{a} \\ &\approx (I + X^\mathfrak{b})(\mathfrak{b} \leftarrow \mathfrak{a})P_\mathfrak{a} \\ &= P_\mathfrak{b} + X^\mathfrak{b} P_\mathfrak{b}. \end{aligned}$$

Hence

$$(D_\mathfrak{b} A_P)(X^\mathfrak{b}) = X^b P_\mathfrak{b}.$$

That is, the (left) derivative of of the group action with respect to the group element is the mapping that takes a tangent vector and "applies its action" to the point. See the last section for why this is intuitive.

It's important to note that this gives a formula for the derivative of the action as a linear mapping, and does not yield a general formula for the Jacobian matrix of that mapping.

## 2. Jacobians

For notational ease we'll drop the frame notation in the below examples. It's assumed that all derivatives are in the output frame.

2.1. **General Jacobian using Generators.** Of course, given the formula $(D_g A_p)(X) = Xgp$, we can trivially compute the Jacobian for any group by choosing a set of generators and applying this map to each. As such, the $i$th column $J_i$ of the Jacobian is given by

$$J_i = G_i gp,$$

where $G_i$ is the (matrix representation of the) $i$th generator.

2.2. **Jacobian of Action of** $\mathrm{SO}(3)$**.** A tangent vector $\omega$ in the Lie algebra of $\mathrm{SO}(3)$ has a matrix representation as the cross product matrix $\omega_\times$. Hence the derivative of the group action $A_p$ by a group element $R$ is given by

$$\begin{aligned}(D_R A_p)(\omega_\times) &= \omega_\times Rp \\ &= \omega \times (Rp) \\ &= (-Rp) \times \omega \\ &= (-Rp)_\times \omega.\end{aligned}$$

Hence the derivative of the group action can be viewed as taking the tangent vector $\omega$ to the point $(-Rp)_\times \omega$, so the Jacobian matrix is $(-Rp)_\times$.

2.3. **Jacobian of Action of** $\mathrm{SE}(3)$**.** A tangent vector in $X \in \mathrm{SE}(3)$ is written as $\begin{bmatrix}\omega \\ v\end{bmatrix}$ where $\omega \in \mathfrak{so}(3)$ is the angular component and $v \in \mathbb{R}^3$ is the linear component. In matrix representation this is

$$X = \begin{bmatrix}\omega_\times & v \\ 0 & 0\end{bmatrix}.$$

Hence the derivative of the action of $g = \begin{bmatrix}R & t \\ 0 & 1\end{bmatrix}$, i.e.

$$A_p : \begin{bmatrix}R & t \\ 0 & 1\end{bmatrix} \mapsto \begin{bmatrix}R & t \\ 0 & 1\end{bmatrix}\begin{bmatrix}p \\ 1\end{bmatrix} = \begin{bmatrix}q \\ 1\end{bmatrix}$$

is given by

$$\begin{aligned}(DA_p)\begin{bmatrix}\omega \\ v\end{bmatrix} &= \begin{bmatrix}\omega_\times & v \\ 0 & 0\end{bmatrix}\begin{bmatrix}q \\ 1\end{bmatrix} \\ &= \begin{bmatrix}\omega \times q + v \\ 0\end{bmatrix} \\ &= \begin{bmatrix}-q \times \omega + v \\ 0\end{bmatrix} \\ &= \begin{bmatrix}-q_\times & I_{3\times3} \\ 0 & 0\end{bmatrix}\begin{bmatrix}\omega \\ v\end{bmatrix}\end{aligned}$$

Dropping the homogeneous coordinate, we see that the Jacobian matrix is $\begin{bmatrix}-q_\times & I_{3\times3}\end{bmatrix}$. Note that you can also derive this matrix by differentiating with respect to the rotation and translation components independently.

## 3. The "Action" of the Lie Algebra

In the first section, we saw that, under the matrix representation of the group and algebra, we have the almost-intuitive-seeming formula for the derivative of the action

$$D_{\mathfrak{b}} A_P(X^{\mathfrak{b}}) = X^{\mathfrak{b}} P_{\mathfrak{b}},$$

which says "the derivative of left-multiplication by the group is left-multiplication by the algebra". It turns out there is a physical interpretation of the Lie algebra that makes this formula meaningful:

Consider a time-parameterized transformation $x(t) = (\mathfrak{b}(t) \leftarrow \mathfrak{a})$, where the output frame $\mathfrak{b}$ is moving with time. At any point in time $t$, the derivative of this transformation (expressed in the moving output frame) is a tangent vector $X^{\mathfrak{b}}(t)$, given by $X^{\mathfrak{b}}(t) = x'(t)x^{-1}(t)$ (the derivative on the right-hand side is coordinate-wise). The output frame is changing, while the input frame is constant, which means we can find an expression for the velocity of points in the output frame under the transformation:

$$\begin{aligned}
\frac{d}{dt} P_{\mathfrak{b}}(t) &= \frac{d}{dt}\left[x(t) P_{\mathfrak{a}}\right] \\
&= x'(t) P_{\mathfrak{a}} \\
&= x'(t) x^{-1}(t) x(t) P_{\mathfrak{a}} \\
&= x'(t) x^{-1}(t) P_{\mathfrak{b}}(t) \\
&= X^{\mathfrak{b}} P_{\mathfrak{b}}(t).
\end{aligned}$$

In short,

$$\frac{d}{dt} P_{\mathfrak{b}}(t) = X^{\mathfrak{b}}(t) P_{\mathfrak{b}}(t).$$

In words, **the derivative of a time-parameterized transformation computes the time derivatives of the acted-on points under the transformation**.

For example, consider SO(3). The output frame time derivative of a rotation $R(t)$ is the skew matrix $\omega_\times = R'(t)R^\top(t)$. This means that the time derivative of an acted-on point $q(t) = R(t)p$ in the ouput frame is

$$q'(t) = \omega \times q(t),$$

which is simply angular velocity.

This interpretation of the Lie algebra makes the notion of "infinitesimal transformation" quite literal: An element of the Lie algebra is a transformation that transforms points infinitesimally, in the sense that it applies a small perturbation (given by the derivative) to each point. It also lets us view the exponential map as the natural solution to an ODE. Namely, say that the transform $x(t)$ has "constant velocity" in the sense that $X^{\mathfrak{b}}$ does not change with time. Then the velocity of points in the output frame is given by

$$\frac{d}{dt} P_{\mathfrak{b}}(t) = X^{\mathfrak{b}} P_{\mathfrak{b}}(t).$$

Since $X^{\mathfrak{b}}$ is constant, there is an exact solution that expresses $P_{\mathfrak{b}}(t)$ at any time in terms of the initial value $P_{\mathfrak{b}}(0)$, namely

$$P_{\mathfrak{b}}(t) = \exp\left(tX^{b}\right) P_{\mathfrak{b}}(0).$$

Hence, for a Lie algebra vector $X^{\mathfrak{b}}$, the exponential transforms points by moving according to the **flow** determined by the velocity field $X^b$ for time 1.